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Stochastic Processes and their Applications 92 (2001) 11–30

stochastic processes and their applications

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Non-linear functionals of the Brownian bridge and some applications ☆

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Received 22 February 2000; received in revised form 7 August 2000; accepted 10 August 2000

Abstract

Let $\{b^F(t), t \in [0,1]\}$ be an F-Brownian bridge process. We study the asymptotic behaviour of non-linear functionals of regularizations by convolution of this process and apply these results to the estimation of the variance of a non-homogeneous diffusion and to the convergence of the number of crossings of a level by the regularized process to a modification of the local time of the Brownian bridge as the regularization parameter goes to 0.

Keywords: Non-linear functionals; Brownian bridge; Regularization by convolution; Crossings; Local time; Non-homogeneous diffusion

1. Introduction

The principal aim of this work is to give a unified setting for the study of the asymptotic behaviour of non-linear functionals of the Brownian bridge and the empirical process. In this direction we prove that apparently unrelated problems such as the asymptotic behaviour of the number of crossings for the density empirical process (defined in Silverman (1978)) and the weak convergence for its L^p-norm (Csörgő and Horváth, 1988) can in fact be treated in the same fashion, via a strong approximation theorem. Indeed both problems can be reduced to the study of some particular non-linear functional of the Brownian bridge for which the convergence will be a direct consequence of a TCL for sums of 1-dependent random variables. We also consider new applications of these results as, for example, to the estimation of the variance of a non-homogeneous diffusion.

☆ Work partially financed by CONICIT, Proyecto de Fortalecimiento de Centros para el Depto. de Matemáticas del IVIC.

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PII: S0304-4149(00)00068-5

We divide our work in two parts. The first one is included in this article where we study the Brownian bridge results. In a second article (Berzin-Joseph et al.), we consider the statistical applications in relation with the empirical process.

Let $\{W(t), t \in [0,1]\}$ be a standard Brownian motion (BM) and φ a convolution kernel. For $\varepsilon > 0$ define the 'regularization' of W by φ as

$$W_\varepsilon(t) = \frac{1}{\varepsilon}\int_{-\infty}^{\infty} \varphi\left(\frac{t-s}{\varepsilon}\right) W(s)\,\mathrm{d}s.$$

Wschebor (1992) showed that if λ denotes Lebesgue's measure on $[0,1]$, then for a.e. ω

$$\lim_{\varepsilon\to 0} \lambda\left\{s \leqslant t\colon \frac{W(s+\varepsilon)-W(s)}{\sqrt{\varepsilon}} \leqslant x\right\} = t\Phi(x)$$

where Φ is the standard Gaussian distribution function. Observe that if $\varphi = \mathbf{1}_{[-1,0]}$ then

$$\frac{W(t+\varepsilon)-W(t)}{\sqrt{\varepsilon}} = \varepsilon^{1/2}\dot{W}_\varepsilon(t)$$

and there is a natural generalization of the previous result: considered as a variable on t over Lebesgue's space, the distribution of $\|\varphi\|_2^{-1}\varepsilon^{1/2}\dot{W}_\varepsilon(t)$ converges in law to a standard Gaussian r.v. This implies that for an a.e. continuous function g with 'moderate' growth,

$$\lim_{\varepsilon\to 0}\int_0^t g\left(\frac{\varepsilon^{1/2}\dot{W}_\varepsilon(s)}{\|\varphi\|_2}\right)\mathrm{d}s = tE[g(N)] \text{ a.s.,}$$

where N is a standard Gaussian r.v.

The speed at which this convergence takes place was studied by Berzin and León (1997):

$$\frac{1}{\sqrt{\varepsilon}}\int_0^t g_1\left(\frac{\varepsilon^{1/2}\dot{W}_\varepsilon(s)}{\|\varphi\|_2}\right)\mathrm{d}s \to \sigma\tilde{W}(t) \quad \text{weakly, as } \varepsilon \to 0$$

where $g_1(x) = g(x) - E[g(N)]$, σ is a positive constant and $\tilde{W}$ is a BM independent of W. This result is useful in the study of the convergence of an integral with respect to the number of crossings of the regularized process. Define $N_\varepsilon^W(x) = \#\{s \leqslant 1 : W_\varepsilon(s) = x\}$, the number of times that the process W_ε crosses level x in $[0,1]$. If φ is absolutely continuous $N_\varepsilon^W(x)$ has finite expectation given by Rice's formula. On the other hand, by a formula of Banach (1925), and Kac (1943),

$$\int_{-\infty}^{\infty} f(x)N_\varepsilon^W(x)\,\mathrm{d}x = \int_0^1 f(W_\varepsilon(s))|\dot{W}_\varepsilon(s)|\,\mathrm{d}s.$$

Wschebor showed that $\lim_{\varepsilon\to 0}\Lambda_\varepsilon\int_{-\infty}^{\infty} f(x)N_\varepsilon^W(x)\,\mathrm{d}x = \int_{-\infty}^{\infty} f(x)\ell^W(x)\,\mathrm{d}x$ a.s., where $\ell^W(x)$ is the Brownian local time at x on $[0,1]$ and $\Lambda_\varepsilon = (\pi\varepsilon/2\|\varphi\|_2^2)^{1/2}$.

In turn, Berzin and León proved that $(1/\sqrt{\varepsilon})\int_{-\infty}^{\infty} f(x)[\Lambda_\varepsilon N_\varepsilon^W(x) - \ell^W(x)]\,\mathrm{d}x \xrightarrow{\mathscr{D}} V$, where V, conditionally on the σ-field $\mathscr{F}$ generated by $\{W(t), t \in [0,1]\}$, is a centred Gaussian r.v. with variance $\sigma^2\int_0^1 f^2(W(s))\,\mathrm{d}s$. This can also be expressed by saying that $\mathscr{L}(V/\mathscr{F}) = \sigma\int_0^1 f(W(s))\,\mathrm{d}\tilde{W}(s)$. Although in that article only weak convergence was considered it is not difficult to see that in fact, stable convergence holds (for the definition see Section 2.1).

To describe the results in this paper we need to introduce some definitions and notation. Let F be an absolutely continuous distribution function with density s. We shall assume that it has bounded support and, without loss of generality, that it is $[0,1]$ i.e. $F(0)=0$ and $F(1)=1$. We also assume that s is continuous and $s(x)>0$ for $x\in[0,1]$. The F-Brownian motion (F-BM) is defined as $W^F(t)=W(F(t))$. The results we have stated can be generalized to this class of processes.

Let $g(x,y)$ be an a.s. continuous function in $L^2(\phi(y)\,\mathrm{d}y)$ with polynomial growth in the second variable that satisfies the following two conditions:
(i) $Eg(x,N)=0$, $0\leqslant x$.
(ii) $E[Ng(x,N)]=0$, $0\leqslant x$.
Define

$$g_\varepsilon(u,y)=g(\sqrt{\varepsilon}\dot{\sigma}_\varepsilon^W(u),y),\quad \tilde{g}_\varepsilon(u,y)=g(\sqrt{\varepsilon}\dot{\sigma}_\varepsilon^b(u),y),$$

$$S_\varepsilon^W(t)=\frac{1}{\sqrt{\varepsilon}}\int_0^t g_\varepsilon(u,\xi_\varepsilon^W(u))\,\mathrm{d}u\quad\text{with}\quad \xi_\varepsilon^W(u)=\frac{\dot{W}_\varepsilon^F(u)}{\dot{\sigma}_\varepsilon^W(u)}$$

and

$$\varepsilon(\dot{\sigma}_\varepsilon^W(u))^2=\varepsilon\,\mathrm{Var}(\dot{W}_\varepsilon^F(u))=\frac{1}{\varepsilon}\int_{-\infty}^{\infty}\varphi^2\left(\frac{u-v}{\varepsilon}\right)s(v)\,\mathrm{d}v\to s(u)\|\varphi\|_2^2. \tag{1}$$

We show in Theorem 2, that, as before, there exists an independent BM $\tilde{W}$ such that $S_\varepsilon^{W^F}(t)\to\tilde{W}(\sigma^2(t))$ stably, where

$$\sigma^2(t)=\int_0^t\int_{-2}^2 E[g(\|\varphi\|_2\sqrt{s(u)},X)g(\|\varphi\|_2\sqrt{s(u)},\theta(w)X+\sqrt{1-\theta^2(w)}Y)]\,\mathrm{d}w\,\mathrm{d}u,$$

X,Y are independent, standard Gaussian variables, $\theta(w)=\psi(w)/\|\varphi\|_2^2$, $\psi(w)=\varphi*\bar{\varphi}(w)$, $\bar{\varphi}(w)=\varphi(-w)$. Stable convergence is considered in Section 2.

Let $\{b^F(t)\}$, be the F-Brownian bridge (F-BB): $b^F(t)=W^F(t)-F(t)W(1)$. We also want to study

$$S_\varepsilon^{b^F}(t)=\frac{1}{\sqrt{\varepsilon}}\int_0^t g_\varepsilon(u,\xi_\varepsilon^{b^F}(u))\,\mathrm{d}u\quad\text{with}\quad \xi_\varepsilon^{b^F}(u)=\frac{\dot{b}_\varepsilon^F(u)}{\dot{\sigma}_\varepsilon^{b^F}(u)}.$$

We show in Theorem 3, that, again, there exists an independent BM $\tilde{W}$ such that $S_\varepsilon^{b^F}(t)\to\tilde{W}(\sigma^2(t))$ stably.

As applications of the previous results we have in Section 4 the following:

(1) The r.v.

$$\frac{1}{E(|N|^\beta)\|\varphi\|_2^\beta}\int_0^t|\sqrt{\varepsilon}\,\dot{b}_\varepsilon^F(u)|^\beta\,\mathrm{d}u,\quad \beta\in\mathbb{R}^+,$$

is an asymptotically unbiased and consistent estimator of $\int_0^t(s(u))^{\beta/2}\,\mathrm{d}u$ (see Section 3).

(2) The following two convergence results hold as $\varepsilon\to0$ (see Section 4.1)

$$\frac{1}{\sqrt{\varepsilon}}\int_0^t\left[\left|\frac{b^F(u+\varepsilon)-b^F(u)}{\sqrt{\varepsilon}}\right|^\beta-E(|N|^\beta)(s(u))^{\beta/2}\right]\mathrm{d}u\xrightarrow{\mathscr{D}}\tilde{W}(\sigma_\beta^2(t)),\quad \beta\geqslant1$$

$$\frac{1}{\sqrt{\varepsilon}}\left[\lambda\left\{u\leqslant t:\frac{b^F(u+\varepsilon)-b^F(u)}{\sqrt{\varepsilon}}\leqslant x\right\}-\int_0^t P(\sqrt{s(u)}N\leqslant x)\,\mathrm{d}u\right]$$

$$\xrightarrow{\mathscr{D}}\tilde{W}(\hat{\sigma}^2(t,x))-K(t,x)$$

where σ_β is defined in the statement of Corollary 3, $\hat{\sigma}(x,t)$ and $K(t,x)$ in Corollary 4.

(3) Let $N_\varepsilon^{b^F}(x)$ be the number of times that the process $b_\varepsilon^F(\cdot)$ crosses level x before time 1 and let $\tilde{\ell}^{b^F}(\cdot)$ be a modification of the local time for the F-BB on $[0,1]$ (see (4) in Section 2 for the definition). We obtain in Section 4.3 the following:

$$\frac{1}{\sqrt{h}}\int_{-\infty}^{\infty} f(x)(\Lambda_\varepsilon N_\varepsilon^{b^F}(x)-\tilde{\ell}^{b^F}(x))\,\mathrm{d}x\to V\quad\text{stably}$$

where V is a centred Gaussian r.v. with random variance $\sigma_1^2\int_0^1 f^2(b^F(u))s(u)\,\mathrm{d}u$, and σ_1 is defined in the next section.

(4) We apply our results to the problem of estimating the variance $\vartheta(t)$ of a non-homogeneous diffusion which has been considered before by Genon-Catalot et al. (1992) and Soulier (1998) who considered a discretization of the diffusion instead of regularization by convolution. Let

$$\mathrm{d}X(t)=\vartheta(t)\,\mathrm{d}W(t)+b(X(t))\,\mathrm{d}t\quad\vartheta(t)>0$$

and suppose that one only observes a regularization by convolution of its solution. We consider the problem of estimating $\vartheta^2(t)$, $\vartheta(t)$ and $\log\vartheta(t)$ in this context.

2. Hypothesis and notation

Let F be a distribution function with bounded support and density s. To simplify the notation we shall suppose that its support is $[0,1]$, i.e. $F(0)=0$ and $F(1)=1$. The F-Brownian motion is defined as $W^F(t)=W(F(t))$, where W is a standard BM. With this definition one has $E(W^F(u)W^F(v))=F(u\wedge v)$.

The F-Brownian bridge is defined as $b^F(t)=W^F(t)-F(t)W(1)$ and then $E(b^F(u)b^F(v))=F(u\wedge v)\,[1-F(u\vee v)]$.

For each t and $\varepsilon>0$ we define the regularized processes $b_\varepsilon^F(t)=\varphi_\varepsilon * b^F(t)$ and $W_\varepsilon^F(t)=\varphi_\varepsilon * W^F(t)$ with $\varphi_\varepsilon(t)=(1/\varepsilon)\varphi(t/\varepsilon)$ where $*$ denotes the convolution.

We shall use the Hermite polynomials, which can be defined by $\exp(tx-t^2/2)=\sum_{n=0}^{\infty}H_n(x)t^n/n!$. They form an orthogonal system for the standard Gaussian measure $\phi(x)\,\mathrm{d}x$ and, if $h\in L^2(\phi(x)\,\mathrm{d}x)$, $h(x)=\sum_{n=0}^{\infty}\hat{h}_nH_n(x)$ and $\|h\|_{2,\phi}^2=\sum_{n=0}^{\infty}n!\,\hat{h}_n^2$. Mehler's formula (Breuer and Major, 1983) gives a simple form to compute the covariance between two L^2 functions of Gaussian r.v.'s: If (X,Y) is a Gaussian random vector having correlation ρ then

$$E[h(X)k(Y)]=\sum_{n=0}^{\infty}\hat{h}_n\hat{k}_nn!\rho^n. \tag{2}$$

We will also use the following well-known property:

$$\int_{-\infty}^{z}H_k(y)\phi(y)\,\mathrm{d}y=-H_{k-1}(z)\phi(z),\quad z\in\mathbb{R}. \tag{3}$$

We have the following hypothesis

(H1) *For the kernel* φ: $\int_{-1}^{1}\varphi(t)\,dt = 1$, $\varphi \geqslant 0, \varphi$, absolutely continuous and the support of φ is a subset of $[-1,1]$. Define $\psi(w) = \varphi * \bar{\varphi}(w)$ where $\bar{\varphi}(w) = \varphi(-w)$ and $\theta(w) = \psi(w)\|\varphi\|_2^{-2}$, $w \in \mathbb{R}$.

(H2) *For the function s*: s is continuous on $[0,1]$ and $0 < s(x)$ for all $x \in [0,1]$.

We shall write

$$l(x) = \sqrt{\frac{\pi}{2}}|x| - 1 = \sum_{n=1}^{\infty} a_{2n}H_{2n}(x) \quad \text{and} \quad \sigma_1^2 = \int_{-2}^{2}\sum_{n=1}^{\infty} a_{2n}^2(2n!)\theta^{2n}(w)\,dw.$$

We have $\dot{b}_\varepsilon^F(t) = (1/\varepsilon)\int_{-\infty}^{t/\varepsilon} b^F(t-\varepsilon y)\,d\varphi(y)$ and a similar expression for $\dot{W}_\varepsilon^F(t)$. Also

$$Z_\varepsilon^{b^F}(f) = \varepsilon^{-1/2}\int_{-\infty}^{\infty} f(x)[\Lambda_\varepsilon N_\varepsilon^{b^F}(x) - \tilde{\ell}^{b^F}(x)]\,dx \quad \text{with } \Lambda_\varepsilon^{-1} = \sqrt{\frac{2}{\pi\varepsilon}}\|\varphi\|_2$$

where $N_\varepsilon^{b^F}(x)$ is the number of times that the process $b_\varepsilon^F(\cdot)$ crosses level x before time 1 and $\tilde{\ell}^{b^F}(\cdot)$ is a modification of the local time for the BB on $[0,1]$ that satisfies, for any continuous function f,

$$\int_{-\infty}^{\infty} f(x)\tilde{\ell}^{b^F}(x)\,dx = \int_0^1 f(b^F(u))\sqrt{s(u)}\,du. \tag{4}$$

Also,

$$(\sigma^b)^2 = \int_0^1 E[f^2(b^F(u))]s(u)\,du, \qquad (\dot{\sigma}_\varepsilon^b(u))^2 = \mathrm{Var}(\dot{b}_\varepsilon^F(u)).$$

For $0 \leqslant t \leqslant 1$ define,

$$S_\varepsilon^{b^F}(t) = \frac{1}{\sqrt{\varepsilon}}\int_0^t g_\varepsilon(u, \xi_\varepsilon^{b^F}(u))\,du \quad \text{where } \xi_\varepsilon^{b^F}(u) = \frac{\dot{b}_\varepsilon^F(u)}{\dot{\sigma}_\varepsilon^b(u)} \quad \text{and by } S_\varepsilon^{W^F}(t)$$

the corresponding integral for W^F.

In what follows we shall drop the indices F when no confusion is possible. Throughout the paper, Const shall stand for a generic constant, whose value may change during a proof. We also use [ ] for the integer part and N denotes a standard Gaussian r.v.

2.1. Stable convergence

We shall use the notion of stable convergence, which we describe now following Aldous and Eagleson (1978), Hall and Heyde (1980) and Jacod (1997). Let X_n be a sequence of r.v.'s defined over $(\Omega, \mathscr{F}, P)$ and taking values in $C[0,1]$, and let $\mathscr{G} \subset \mathscr{F}$ be a sub-σ-field. Let X be another r.v. defined over an extension $(\bar{\Omega}, \bar{\mathscr{F}}, \bar{P})$ of the original probability space, with values in $C[0,1]$. The sequence X_n converges $\mathscr{G}$-stably to X if

$$\lim_n E(Zh(X_n)) = \bar{E}(Zh(X)), \tag{5}$$

for all bounded and continuous functions $h: C[0,1] \to \mathbb{R}$ and all $\mathscr{G}$-measurable and bounded r.v. Z.

Stable convergence is invariant under absolutely continuous changes of the measure.

3. Results

It is easy to see, using the properties of W and φ, that

$$\Gamma_\varepsilon^b(u,v) := \varepsilon E(\dot{b}_\varepsilon^F(u)\dot{b}_\varepsilon^F(v))$$
$$= \int_{-\infty}^{+\infty} \varphi(w)\varphi\left(\frac{v-u}{\varepsilon}+w\right)s(u-\varepsilon w)\,\mathrm{d}w$$
$$- \varepsilon\left[\int_{-\infty}^{+\infty}\varphi(w)s(v-\varepsilon w)\,\mathrm{d}w\right]\left[\int_{-\infty}^{+\infty}\varphi(w)s(u-\varepsilon w)\,\mathrm{d}w\right]. \tag{6}$$

$$\Gamma_\varepsilon^W(u,v) := \varepsilon E(\dot{W}_\varepsilon^F(u)\dot{W}_\varepsilon^F(v)) = \int_{-\infty}^{+\infty}\varphi(w)\varphi\left(\frac{v-u}{\varepsilon}+w\right)s(u-\varepsilon w)\,\mathrm{d}w. \tag{7}$$

$$\gamma_\varepsilon^W(u,v) := E(W_\varepsilon^F(u)W_\varepsilon^F(v)) = \int_{-\infty}^{+\infty}\int_{-\infty}^{(v-u)/\varepsilon+w}\varphi(w)\varphi(z)F(u-\varepsilon w)\,\mathrm{d}z\,\mathrm{d}w$$
$$+ \int_{-\infty}^{+\infty}\int_{-\infty}^{(u-v)/\varepsilon+w}\varphi(w)\varphi(z)F(v-\varepsilon w)\,\mathrm{d}z\,\mathrm{d}w. \tag{8}$$

Observe that if $|u-v| \geqslant 2\varepsilon$ then $\Gamma_\varepsilon^W(u,v) = 0$.

A simple calculation shows that if $s(u)$ is continuous and $s(u) > 0, 0 < u < 1$, the law of the F-Brownian bridge $b^F(t)$ $(t < 1)$ is absolutely continuous with respect to the F-Brownian motion $W^F(t)$ and its density is

$$\frac{1}{\sqrt{1-F(t)}}\exp\left\{-\frac{(W(F(t))^2}{2(1-F(t))}\right\}.$$

Theorem 1. *Under* (H1) *and* (H2)

$$\lambda\{u \leqslant t\colon \sqrt{\varepsilon}\,\dot{b}_\varepsilon^F(u) \leqslant x\} \to \int_0^t P(\sqrt{s(u)}\|\varphi\|_2 N \leqslant x)\,\mathrm{d}u, \quad \text{a.s. as } \varepsilon \to 0.$$

Proof. Consider the set

$$\Delta_b = \left\{\omega\colon \lim_{\varepsilon\to 0}\lambda\{u \leqslant t\colon \sqrt{\varepsilon}\dot{b}_\varepsilon^F(u,\omega) \leqslant x\} = \int_o^t P(\sqrt{s(u)}\|\varphi\|_2 N \leqslant x)\,\mathrm{d}u\right\}$$

and define Δ_W similarly with $\dot{W}$ instead of $\dot{b}$. Then

$$\int_\Omega \mathbf{1}_{\Delta_b}(\omega)\,\mathrm{d}\mu_{b^F}(\omega) = \int_\Omega \mathbf{1}_{\Delta_W}(\omega)\frac{\mathrm{e}^{-W^2(F(t))/2(1-F(t))}}{\sqrt{1-F(t)}}\mathrm{d}\mu_{W^F}(\omega)$$

where $\mathrm{d}\mu_{b^F}$ and $\mathrm{d}\mu_{W^F}$ are the measures corresponding to b^F and W^F and Ω is the probability space where both processes live. Since W^F is obtained by a deterministic change of time from W, by Wschebor (1992) we know that

$$P(\Delta_W) = 1$$

thus the previous integral is

$$\int_\Omega \frac{\mathrm{e}^{-W^2(F(t))/2(1-F(t))}}{\sqrt{1-F(t)}}\,\mathrm{d}\mu_{W^F}(\omega) = \int_\Omega \mathrm{d}\mu_{b^F}(\omega) = 1. \qquad \square$$

Corollary 1. *For every $\beta \geqslant 0$, as $\varepsilon \to 0$*

$$\int_0^t |\sqrt{\varepsilon}\,\dot{b}_\varepsilon^F(u)|^\beta \,\mathrm{d}u \to E(|N|^\beta)\|\varphi\|_2^\beta \int_0^t [s(u)]^{\beta/2}\,\mathrm{d}u \quad \text{a.s.}$$

We have, thus, an asymptotically unbiased and consistent estimator of $\int_0^t [\sqrt{s(u)}]^\beta \,\mathrm{d}u$ given by

$$\frac{1}{E(|N|^\beta)\|\varphi\|_2^\beta}\int_0^t |\sqrt{\varepsilon}\,\dot{b}_\varepsilon^F(u)|^\beta \,\mathrm{d}u.$$

Corollary 2. *For f continuous*

$$\Lambda_\varepsilon \int_{-\infty}^{\infty} f(x)N_\varepsilon^b(x)\,\mathrm{d}x \to \int_{-\infty}^{\infty} f(x)\tilde{\ell}^b(x)\,\mathrm{d}x \quad a.s. \quad as \quad \varepsilon \to 0.$$

Proof. Recall that

$$\Lambda_\varepsilon = \left(\frac{\pi\varepsilon}{2\|\varphi\|_2^2}\right)^{1/2}.$$

As in Wschebor (1992) we deduce from Corollary 1 that a.s. for every continuous $h:[0,1] \to \mathbb{R}$, as $\varepsilon \to 0$

$$\int_0^1 \sqrt{\varepsilon}|\dot{b}_\varepsilon^F(u)|\,h(u)\,\mathrm{d}u \to \left(\frac{2}{\pi}\right)^{1/2} \|\varphi\|_2 \int_0^1 \sqrt{s(u)}\,h(u)\,\mathrm{d}u. \tag{9}$$

On the other hand, for $f: \mathbb{R} \to \mathbb{R}$ and g of class C^1, $\int_{-\infty}^{\infty} f(x)N^g(x)\,\mathrm{d}x = \int_0^1 f(g(t))|g'(t)|\,\mathrm{d}t$ where $N^g(x)$ is the number of crossings of x by g on $[0,1]$ (see Banach, 1925; Kac, 1943). Hence

$$\int_{-\infty}^{\infty} f(x)\sqrt{\varepsilon}N_\varepsilon^b(x)\,\mathrm{d}x = \int_0^1 [f(b_\varepsilon^F(u)) - f(b^F(u))]\sqrt{\varepsilon}|\dot{b}_\varepsilon^F(u)|\,\mathrm{d}u$$
$$+ \int_0^1 f(b^F(u))\sqrt{\varepsilon}|\dot{b}_\varepsilon^F(u)|\,\mathrm{d}u$$

The continuity of f, the uniform convergence of $b_\varepsilon^F(t)$ to $b^F(t)$, the boundedness of $b^F(t)$ and $b_\varepsilon^F(t)$ for $t \in [0,1]$ and the fact that $\int_0^1 \sqrt{\varepsilon}|\dot{b}_\varepsilon^F(u)|\,\mathrm{d}u$ is a.s. bounded imply the convergence to 0 of the first term as $\varepsilon \to 0$. The limit of the second term can be obtained from (9). □

Let $g(x,y)$, $x \in \mathbb{R}^+, y \in \mathbb{R}$, be an a.s. continuous function in $L^2(\phi(y)\mathrm{d}y)$ continuous in the first variable and with polynomial growth in the second variable ($g(x,y) \leqslant KP(|y|)$), that satisfies the following two conditions:

(i) $Eg(x,N) = 0$, $0 \leqslant x$.

(ii) $E[Ng(x,N)] = 0$, $0 \leqslant x$.

In what follows, the first variable x belongs to a compact set and therefore the function is bounded in the first variable. Let $g(x,y) = \sum_{k=2}^{\infty} c_k(x)H_k(y)$ be the Hermite expansion of g and define

$$g_\varepsilon(u,y) = g(\sqrt{\varepsilon}\;\dot{\sigma}_\varepsilon^W(u), y), \quad \text{and} \quad \tilde{g}_\varepsilon(u,y) = g(\sqrt{\varepsilon}\;\dot{\sigma}_\varepsilon^b(u), y)$$

$$S_\varepsilon^W(t) = \frac{1}{\sqrt{\varepsilon}} \int_0^t g_\varepsilon(u, \xi_\varepsilon^W(u))\,\mathrm{d}u \quad \text{with} \quad \xi_\varepsilon^W(u) = \frac{\dot{W}_\varepsilon^F(u)}{\dot{\sigma}_\varepsilon^W(u)}$$

and

$$S_\varepsilon^b(t) = \frac{1}{\sqrt{\varepsilon}} \int_0^t \tilde{g}_\varepsilon(u, \xi_\varepsilon^b(u))\,\mathrm{d}u \quad \text{with} \quad \xi_\varepsilon^b(u) = \frac{\dot{b}_\varepsilon^F(u)}{\dot{\sigma}_\varepsilon^b(u)}.$$

Using (1), observe that $g_\varepsilon(u, y) \to g(\|\varphi\|_2\sqrt{s(u)}, y)$ as $\varepsilon \to 0$ for *a.e.* $u \in [0,1]$, and that the same is true for $\tilde{g}_\varepsilon(u, y)$. Define

$$\sigma^2(t) = \int_0^t h(u)\,\mathrm{d}u,$$

where

$$h(u) = \sum_{k=2}^{\infty} k!\,c_k^2(\|\varphi\|_2\sqrt{s(u)}) \left[\int_{-2}^{2} \theta^k(w)\,\mathrm{d}w\right]$$

$$= \int_{-2}^{2} E[g(\|\varphi\|_2\sqrt{s(u)}, X) g(\|\varphi\|_2\sqrt{s(u)}, \theta(w)X + \sqrt{1-\theta^2(w)}Y)]\,\mathrm{d}w,$$

X, Y are independent, standard Gaussian variables, h is continuous and non-negative in $[0,1]$.

For the next theorem we need a Brownian motion independent of W. This can be constructed as in Jacod (1997): Let $\mathscr{F}_t$ be the σ-field generated by W_s, $0 \leqslant s \leqslant t$ and set $\mathscr{F} = \bigvee \mathscr{F}_t, t \leqslant 1$. Consider an extension of the original filtered space $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{\mathscr{F}}_t, \tilde{P})$, such that there exists a Wiener process $\tilde{W}$ with respect to a filtration $\mathscr{G}_t$ (sub-σ-field of $\mathscr{F}_t$), which is independent of the original process W. This can be done by defining $\tilde{W}$ as the canonical process on the canonical space $(\Omega_1, \mathscr{G}, \mathscr{G}_t, P_1)$ and setting $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{\mathscr{F}}_t)$ to be the product of $(\Omega, \mathscr{F}, \mathscr{F}_t)$ by $(\Omega_1, \mathscr{G}, \mathscr{G}_t)$, $\tilde{P}(\mathrm{d}\omega, \mathrm{d}\omega_1) = P(\mathrm{d}\omega)P_1(\mathrm{d}\omega_1)$.

Theorem 2. *Under conditions* (i) *and* (ii)

$$(W_\varepsilon^F(\cdot), S_\varepsilon^{W^F}(\cdot)) \to (W^F(\cdot), \tilde{W}(\sigma^2(\cdot))) \quad \textit{stably}$$

where $\tilde{W}$ is a BM independent of W.

Proof. Using conditions (i) and (ii) g_ε has Hermite expansion

$$g_\varepsilon(u, y) = \sum_{k=2}^{\infty} c_{k,\varepsilon}(u) H_k(y)$$

and

$$c_{k,\varepsilon}(u) \to c_k(\|\varphi\|_2\sqrt{s(u)}) = \frac{1}{k!} \int_{-\infty}^{\infty} g(\|\varphi\|_2\sqrt{s(u)}, y)\,\phi(y) H_k(y)\,\mathrm{d}y < \infty$$

$$\text{for } a.e.\ u \in [0,1] \tag{10}$$

We may assume as in Berzin-Joseph and León (1997) that $t > 0$ and therefore there exists an ε such that $t \geqslant 4\varepsilon$, hence

$$E[S_\varepsilon^{W^F}(t)]^2 \simeq \frac{1}{\varepsilon} \int_0^t \int_0^t \sum_{k=2}^{\infty} c_{k,\varepsilon}(u) c_{k,\varepsilon}(v) E[H_k(\xi_\varepsilon^W(u)) H_k(\xi_\varepsilon^W(v))] \mathbf{1}_{\{2\varepsilon \leqslant u \leqslant t-2\varepsilon\}}$$
$$\times \mathbf{1}_{\{2\varepsilon \leqslant v \leqslant t-2\varepsilon\}}\,\mathrm{d}v\,\mathrm{d}u,$$

where $\simeq$ means asymptotically equivalent. Using Mehler's formula (2) and making $v = u + \varepsilon\omega$ we obtain

$$E[S_\varepsilon^W(t)]^2$$
$$\simeq \int_{2\varepsilon}^{t-2\varepsilon} \int_{\frac{-u}{\varepsilon}}^{(t-u)/\varepsilon} \sum_{k=2}^{\infty} k! c_{k,\varepsilon}(u) c_{k,\varepsilon}(u+\varepsilon w)(E[\xi_\varepsilon^W(u)\xi_\varepsilon^W(u+\varepsilon w)])^k$$
$$\times \mathbf{1}_{\{2\varepsilon \leqslant u+\varepsilon w \leqslant t-2\varepsilon\}} \, dw \, du.$$

We split the integral into three:

$$J_1 = \int_{2\varepsilon}^{t-2\varepsilon} \int_{-2}^{2} \sum_{k=2}^{\infty} k! c_{k,\varepsilon}(u) c_{k,\varepsilon}(u+\varepsilon w)(E[\xi_\varepsilon^W(u)\xi_\varepsilon^W(u+\varepsilon w)])^k$$
$$\times \mathbf{1}_{\{2\varepsilon \leqslant u+\varepsilon w \leqslant t-2\varepsilon\}} \, dw \, du,$$

$$J_2 = \int_{2\varepsilon}^{t-2\varepsilon} \int_{2}^{(t-u)/\varepsilon} \sum_{k=2}^{\infty} k! c_{k,\varepsilon}(u) c_{k,\varepsilon}(u+\varepsilon w)(E[\xi_\varepsilon^W(u)\xi_\varepsilon^W(u+\varepsilon w)])^k$$
$$\times \mathbf{1}_{\{2\varepsilon \leqslant u+\varepsilon w \leqslant t-2\varepsilon\}} \, dw \, du,$$

$$J_3 = \int_{2\varepsilon}^{t-2\varepsilon} \int_{-u/\varepsilon}^{-2} \sum_{k=2}^{\infty} k! c_{k,\varepsilon}(u) c_{k,\varepsilon}(u+\varepsilon w)(E[\xi_\varepsilon^W(u)\xi_\varepsilon^W(u+\varepsilon w)])^k$$
$$\times \mathbf{1}_{\{2\varepsilon \leqslant u+\varepsilon w \leqslant t-2\varepsilon\}} \, dw \, du.$$

By the independence of the increments we have $J_2 = J_3 = 0$ and we only have to consider J_1, but by (7) $(E[\xi_\varepsilon^W(u)\xi_\varepsilon^W(u+\varepsilon w)])^k \overset{\varepsilon\to 0}{\to} \theta^k(w)$, and by (10), $c_{k,\varepsilon}(u)c_{k,\varepsilon}(u+\varepsilon w) \overset{\varepsilon\to 0}{\to} c_k^2(\|\varphi\|_2\sqrt{s(u)})$ for a.e. $u \in [0,1]$.

Now

$$|c_{k,\varepsilon}(u)c_{k,\varepsilon}(u+\varepsilon\omega)k!(E[\xi_\varepsilon^W(u)\xi_\varepsilon^W(u+\varepsilon w)])^k|\mathbf{1}_{\{2\varepsilon\leqslant u\leqslant t-2\varepsilon\}}\mathbf{1}_{\{2\varepsilon\leqslant u+\varepsilon w\leqslant t-2\varepsilon\}}$$
$$\times \mathbf{1}_{\{-2\leqslant w\leqslant 2\}} \leqslant |c_{k,\varepsilon}(u)c_{k,\varepsilon}(u+\varepsilon\omega)|k!$$

and also

$$\sum_{k=2}^{\infty} |c_{k,\varepsilon}(u)c_{k,\varepsilon}(u+\varepsilon\omega)|k! \leqslant \left(\sum_{k=2}^{\infty} c_{k,\varepsilon}^2(u)k!\right)^{1/2} \left(\sum_{k=2}^{\infty} c_{k,\varepsilon}^2(u+\varepsilon w)k!\right)^{1/2}$$
$$= \|g_\varepsilon(u,\cdot)\|_2 \; \|g_\varepsilon(u+\varepsilon w,\cdot)\|_2$$

and this is bounded uniformly by the definition of g_ε and the continuity of g and its polynomial growth in the second variable. Hence we can interchange limits and integrals. Taking into account that $\sum_{k=2}^{\infty} c_{k,\varepsilon}^2(u)k!$ and $\sum_{k=2}^{\infty} c_{k,\varepsilon}^2(u+\varepsilon w)k!$ both converge to the same limit $\int_{-\infty}^{\infty} g^2(\|\varphi\|_2\sqrt{s(u)}, y)\phi(y)\,dy = \sum_{k=2}^{\infty} c_k^2(\|\varphi\|_2\sqrt{s(u)})k!$ for a.e. $u \in [0,1]$ we obtain

$$J_1 \to \int_0^t \int_{-2}^{2} \sum_{k=2}^{\infty} k! c_k^2(\|\varphi\|_2\sqrt{s(u)})\,\theta^k(w)\,dw\,du = \int_0^t h(u)\,du = \sigma^2(t) \quad \text{as } \varepsilon \to 0.$$

We also get

$$E[S_\varepsilon^W(t)] = \frac{1}{\sqrt{\varepsilon}}\int_0^t E[g_\varepsilon(u,\xi_\varepsilon^W(u))]\,\mathrm{d}u = \frac{1}{\sqrt{\varepsilon}}\int_0^t \sum_{k=2}^{\infty} c_{k,\varepsilon}(u)E[H_k(\xi_\varepsilon^W(u))]\,\mathrm{d}u = 0.$$

The process $S_\varepsilon^W(t)$ has asymptotically independent increments. If we show that

$$E(S_\varepsilon^W(t) - S_\varepsilon^W(s))^4 \leqslant \mathrm{Const}|t-s|^2 \tag{11}$$

this will imply the uniform integrability of $(S_\varepsilon^W(t))^2$ and the tightness of $S_\varepsilon^W(\cdot)$. Let us check (11).

$$S_\varepsilon^W(t) - S_\varepsilon^W(s) = \frac{1}{\sqrt{\varepsilon}}\int_s^t g_\varepsilon(u,\xi_\varepsilon^W(u))\,\mathrm{d}u = \frac{1}{\sqrt{\varepsilon}}\sum_{k=0}^{N(\varepsilon)-1} Z_\varepsilon(k) + P_{t,s}^\varepsilon$$

where

$$N(\varepsilon) = \left[\frac{t-s}{2\varepsilon}\right],$$

$$Z_\varepsilon(k) = \int_{s+2k\varepsilon}^{s+2(k+1)\varepsilon} g_\varepsilon(u,\xi_\varepsilon^W(u))\,\mathrm{d}u \quad \text{and} \quad P_{t,s}^\varepsilon = \frac{1}{\sqrt{\varepsilon}}\int_{s+2\varepsilon N(\varepsilon)}^t g_\varepsilon(u,\xi_\varepsilon^W(u))\,\mathrm{d}u.$$

It is easy to verify that the $Z_\varepsilon(k)$ are 1-dependent.

(A) Let us calculate first $E(P_{t,s}^\varepsilon)^4$. Using Jensen's inequality and the polynomial growth of g we obtain

$$E(P_{t,s}^\varepsilon)^4 \leqslant \frac{1}{\varepsilon^2}(t-s-2\varepsilon N(\varepsilon))^3 \int_{s+2\varepsilon N(\varepsilon)}^t E[g_\varepsilon^4(u,\xi_\varepsilon^W(u))]\,\mathrm{d}u \leqslant \mathrm{Const}(t-s)^2.$$

(B) Now we calculate

$$E\left[\frac{1}{\sqrt{\varepsilon}}\sum_{k=0}^{N(\varepsilon)-1} Z_\varepsilon(k)\right]^4.$$

Using the Hermite expansion for $g_\varepsilon(u,y)$ in the second variable we deduce that $E(Z_\varepsilon(k)) = 0$. Then we have

$$E\left[\frac{1}{\sqrt{\varepsilon}}\sum_{k=0}^{N(\varepsilon)-1} Z_\varepsilon(k)\right]^4 = \frac{1}{\varepsilon^2}\sum_{k_1,k_2,k_3,k_4} E[Z_\varepsilon(k_1)Z_\varepsilon(k_2)Z_\varepsilon(k_3)Z_\varepsilon(k_4)].$$

We can assume, without loss of generality, that $k_1 \leqslant k_2 \leqslant k_3 \leqslant k_4$. We consider several cases.

(i) If $k_4 - k_3 \geqslant 2$, $Z_\varepsilon(k_4)$ is independent of the other three r.v. and

$$E[Z_\varepsilon(k_1)Z_\varepsilon(k_2)Z_\varepsilon(k_3)Z_\varepsilon(k_4)] = 0.$$

(ii) If $0 \leqslant k_4 - k_3 \leqslant 1$ (here k_4 is a function of k_3), two cases are possible:

(iia) If $k_3 - k_2 \geqslant 1$, $(Z_\varepsilon(k_3), Z_\varepsilon(k_4))$ is independent of $(Z_\varepsilon(k_1), Z_\varepsilon(k_2))$, and

$$E[Z_\varepsilon(k_1)Z_\varepsilon(k_2)Z_\varepsilon(k_3)Z_\varepsilon(k_4)] = E[Z_\varepsilon(k_1)Z_\varepsilon(k_2)]E[Z_\varepsilon(k_3)Z_\varepsilon(k_4)]$$

(iia-α) If $k_2 - k_1 \geqslant 0$, $Z_\varepsilon(k_2)$ is independent of $Z_\varepsilon(k_1)$ and the expectation is zero.

(iia-β) Otherwise $0 \leqslant k_2 - k_1 \leqslant 1$ (we have now a relationship between k_2 and k_1), and the sum has only two independent indices. Using Schwarz's inequality, the a.s. continuity in u and the polynomial growth on y of $g(u,y)$ we get

$$E(Z_\varepsilon^{2p}(k)) \leqslant \text{ Const } \varepsilon^{2p}$$

This implies that the sum is bounded by $\text{Const}\,\varepsilon^2 N^2(\varepsilon) \leqslant \text{Const}(t-s)^2$.

(iib) If $0 \leqslant k_3 - k_2 \leqslant 1$ (we have again a relationship between k_3 and k_2) using

$$E[Z_\varepsilon(k_1)Z_\varepsilon(k_2)Z_\varepsilon(k_3)Z_\varepsilon(k_4)] \leqslant \prod_{i=1}^{4}[E(Z_\varepsilon^4(k_i))]^{1/4}$$

and summing over the two independent indices, we obtain that the sum is bounded by $\text{Const}(t-s)^2$. Using all these results we get (11).

To show the weak convergence, let X be a limit point for the sequence $S_\varepsilon^W(\cdot)$. The process X is continuous and has independent increments. Furthermore it is not difficult to see, using uniform integrability, that $E(X(t)) = 0$ and $E(X^2(t)) = \sigma^2(t)$. Hence by using the properties of $\sigma^2(t)$ and $h(t)$ and Theorem 19.1 of Billingsley (1968) it follows that $X(\cdot) = \tilde{W}(\sigma^2(\cdot))$, where $\tilde{W}$ is a BM.

We have shown that $S_\varepsilon^W(\cdot)$ converges weakly to a BM $\tilde{W}(\sigma^2(\cdot))$. We shall prove that the vector process $(W_\varepsilon^F(\cdot), S_\varepsilon^W(\cdot))$ converges weakly towards $(W^F(\cdot), \tilde{W}(\sigma^2(\cdot)))$ and that the processes $W^F(\cdot)$ and $\tilde{W}(\sigma^2(\cdot))$ are independent.

We have that $W_\varepsilon^F(\cdot) \to W^F(\cdot)$ a.s. and $S_\varepsilon^W(\cdot) \to \tilde{W}(\sigma^2(\cdot))$ weakly. Then the sequence of vector processes $(W_\varepsilon^F(\cdot), S_\varepsilon^W(\cdot))$ is tight in $C[0,1] \times C[0,1]$. Let us prove the independence of the increments. Let $t_1 < t_2 < t_3 < t_4$ and consider the vectors $(W_\varepsilon^F(t_2) - W_\varepsilon^F(t_1), S_\varepsilon^W(t_2) - S_\varepsilon^W(t_1))$ and $(W_\varepsilon^F(t_4) - W_\varepsilon^F(t_3), S_\varepsilon^W(t_4) - S_\varepsilon^W(t_3))$. We can suppose, without loss of generality, that $t_3 - t_2 > 3\varepsilon$ and $t_1 > \varepsilon$.

To study the independence between these two vectors, observe that the first one is in $\mathscr{F}_{t_2+\varepsilon} = \sigma\{W^F(s): s \leqslant t_2 + \varepsilon\}$. Furthermore, if $W^F(s)$ is measurable with respect to this σ-algebra it holds, by independence of the increments, that

$$E[W^F(s)[c_1(W_\varepsilon^F(t_4) - W_\varepsilon^F(t_3)) + c_2\dot{W}_\varepsilon^F(t)]] = 0$$

where $t_3 - \varepsilon \leqslant t$. This fact implies the independence between $\mathscr{F}_{t_2+\varepsilon}$ and the σ-algebra generated by the Gaussian vectors $(W_\varepsilon^F(t_4) - W^F(t_3), \dot{W}_\varepsilon^F(t), t_3 - \varepsilon \leqslant t)$. Given that $(W_\varepsilon^F(t_4) - W_\varepsilon^F(t_3), S_\varepsilon^W(t_4) - S_\varepsilon^W(t_3))$ is measurable with respect to this σ-algebra, the mutual independence holds.

Let Y be a limit point of the sequence $(W_\varepsilon^F(\cdot), S_\varepsilon^W(\cdot))$. Given the previous results, Y is a continuous random vector process having independent increments and finite second moment. Thus it must be Gaussian. If we prove that

$$E(W_\varepsilon^F(t)S_\varepsilon^W(t')) = 0 \tag{12}$$

it follows that $E(W^F(t)\tilde{W}(\sigma^2(t'))) = 0$. Let us see (12).

$$E(W_\varepsilon^F(t)S_\varepsilon^W(t')) = \sqrt{\frac{\gamma_\varepsilon^W(t,t)}{\varepsilon}}E\left[H_1\left(\frac{W_\varepsilon^F(t)}{\sqrt{\gamma_\varepsilon^W(t,t)}}\right)\int_0^{t'} g_\varepsilon(u, \xi_\varepsilon^W(u))\,\mathrm{d}u\right].$$

Using the hypothesis and the polynomial bound for g_ε we get

$$E(W_\varepsilon^F(t)S_\varepsilon^W(t'))$$
$$=\sqrt{\frac{\gamma_\varepsilon^W(t,t)}{\varepsilon}}\int_0^{t'}\sum_{k=2}^{\infty}c_{k,\varepsilon}(u)E\left[H_1\left(\frac{W_\varepsilon^F(t)}{\sqrt{\gamma_\varepsilon^W(t,t)}}\right)H_k(\xi_\varepsilon^W(u))\right]\,\mathrm{d}u=0$$

This shows weak convergence. To prove stable convergence observe that condition (5) is valid if Z is an indicator r.v. and a classical approximation argument shows that it holds in fact for any Z. This finishes the proof of Theorem 2. □

Theorem 3. *Under the same conditions as Theorem* 2,

$$S_\varepsilon^b(t)\to\tilde{W}(\sigma^2(t))\quad\textit{stably}$$

in $C[0,1]$ *where* $\tilde{W}$ *is a Brownian Motion. Furthermore the vector process* $(b_\varepsilon^F(\cdot),$ $S_\varepsilon^b(\cdot))$ *converges weakly in* $C[0,1]\times C[0,1]$ *towards* $(b^F(\cdot),\tilde{W}(\sigma^2(\cdot)))$ *and the processes* $b^F(\cdot)$ *and* $\tilde{W}(\sigma^2(\cdot))$ *are independent.*

Proof. Let

$$\boldsymbol{b}_\varepsilon^F(\boldsymbol{t})=(b_\varepsilon^F(t_1),b_\varepsilon^F(t_2),\dots,b_\varepsilon^F(t_n))$$

and

$$\boldsymbol{S}_\varepsilon^{b^F}(\boldsymbol{t})=(S_\varepsilon^{b^F}(t_1),S_\varepsilon^{b^F}(t_2),\dots,S_\varepsilon^{b^F}(t_n)).$$

To study the stable convergence of the finite-dimensional distributions let G be a bounded continuous function of $2n$ variables and consider

$$E[G(\boldsymbol{b}_\varepsilon^F(\boldsymbol{t}),\boldsymbol{S}_\varepsilon^{b^F}(\boldsymbol{t}))]$$
$$=E\left[G(W_\varepsilon^F(t_1),\dots,W_\varepsilon^F(t_n),S_\varepsilon^{W^F}(t_1),\dots,S_\varepsilon^{W^F}(t_n))\right.$$
$$\left.\times\exp\left\{\frac{-(W^F(t_n+\varepsilon))^2}{2(1-F(t_n+\varepsilon))}\right\}\frac{1}{\sqrt{1-F(t_n+\varepsilon)}}\right]$$

this expression converges to

$$E\left[G(W^F(t_1),\dots,W^F(t_n),\tilde{W}(t_1),\dots,\tilde{W}(t_n))\exp\left\{\frac{-(W^F(t_n))^2}{2(1-F(t_n))}\right\}\frac{1}{\sqrt{1-F(t_n)}}\right]$$
$$=E[G(b^F(t_1),\dots,b^F(t_n),\tilde{W}(t_1),\dots,\tilde{W}(t_n))]$$

Let us consider now the tightness of $(b_\varepsilon^F(t),S_\varepsilon^{b^F}(t))$. We know that the first coordinate converges a.s. to $b^F(t)$ and hence is tight. For the second we calculate the fourth-order moment of an increment

$$E[S_\varepsilon^{b^F}(t_1)-S_\varepsilon^{b^F}(t_2)]^4.$$

Fix $t<1$ and consider the relative position of t_1 and t_2 respect to t. If both points are to the left, we have

$$E[S_\varepsilon^{b^F}(t_1)-S_\varepsilon^{b^F}(t_2)]^4 = E\left[[S_\varepsilon^{W^F}(t_1)-S_\varepsilon^{W^F}(t_2)]^4 \exp\left\{\frac{-(W^F(t))^2}{2(1-F(t))}\right\}\frac{1}{\sqrt{1-F(t)}}\right]$$

$$\leqslant \mathrm{Const}\frac{1}{\sqrt{1-F(t)}}(t_1-t_2)^2.$$

If both are to the right, we consider the process with reversed time from 1: $b^F(1-t)$, which is also a bridge, and the previous proof holds. If $t_2<t<t_1$ then consider the increments over the intervals (t_2,t) and (t,t_1) and apply the previous calculations to each one to obtain

$$\begin{aligned} E[S_\varepsilon^{b^F}(t_1)-S_\varepsilon^{b^F}(t_2)]^4 &\leqslant C_1\{E[S_\varepsilon^{b^F}(t_1)-S_\varepsilon^{b^F}(t)]^4+E[S_\varepsilon^{b^F}(t)-S_\varepsilon^{b^F}(t_2)]^4\} \\ &\leqslant C_1\{(t-t_2)^2+(t_1-t)^2\} \\ &\leqslant C_2(t_1-t_2)^2. \end{aligned}$$

These inequalities show that the finite-dimensional distributions also converge for $t=1$. □

4. Applications

4.1. Increments

Corollary 3. *Under* (H2) *and if* $s\in C^2[0,1]$, *for every* $\beta\geqslant 1$

$$\frac{1}{\sqrt{\varepsilon}}\int_0^t\left[\left|\frac{b^F(u+\varepsilon)-b^F(u)}{\sqrt{\varepsilon}}\right|^\beta - E|N|^\beta[s(u)]^{\beta/2}\right]\mathrm{d}u \xrightarrow{\mathscr{D}} \tilde{W}(\sigma_\beta^2(t))$$

in $C[0,1]$ *as* $\varepsilon\to 0$ *where* N *is a standard Gaussian variable and*

$$\sigma_\beta^2(t)=2\left[\int_0^t (s(u))^\beta\,\mathrm{d}u\right]\left[\sum_{k=1}^\infty \frac{1}{(2k+1)!}\left[\int_{-\infty}^{\infty}|y|^\beta\phi(y)H_{2k}(y)\,\mathrm{d}y\right]^2\right].$$

Proof. Observe first that

$$\frac{b^F(u+\varepsilon)-b^F(u)}{\sqrt{\varepsilon}}=\sqrt{\varepsilon}\,\dot{b}_\varepsilon^F(u)$$

when $\varphi=\mathbf{1}_{[-1,0]}$. Let us show that

$$\begin{aligned} Z_\varepsilon(t) &= \frac{1}{\sqrt{\varepsilon}}\int_0^t\left[|\sqrt{\varepsilon}\,\dot{b}_\varepsilon^F(u)|^\beta - E|N|^\beta|\sqrt{\varepsilon}\,\dot{\sigma}_\varepsilon^b(u)|^\beta\right]\mathrm{d}u \\ &= \frac{1}{\sqrt{\varepsilon}}\int_0^t g_\varepsilon(u,\xi_\varepsilon^b(u))\,\mathrm{d}u \to \tilde{W}(\sigma_\beta^2(t)) \end{aligned}$$

weakly as $\varepsilon\to 0$, where $g(x,y)=x(|y|^\beta-E|N|^\beta)$.

Since $s(u)$ is continuous, $\theta(u)=(1-|u|)\mathbf{1}_{\{-1\leqslant u\leqslant 1\}}$ and $\int_{-1}^{1}[\theta(u)]^k\,\mathrm{d}u=2/(k+1)$,

$$h(u)=\sum_{k=1}^{\infty}(s(u))^{\beta}\frac{2}{(2k+1)!}\left[\int_{-\infty}^{\infty}|y|^{\beta}H_{2k}(y)\,\phi(y)\,\mathrm{d}y\right]^2$$

is continuous for $u\in[0,1]$ and it is strictly positive in $[0,1]$. To finish the proof of Corollary 3 let us see that

$$\frac{1}{\sqrt{\varepsilon}}\int_0^t(|\sqrt{\varepsilon}\dot{\sigma}_{\varepsilon}^b(u)|^{\beta}-(s(u))^{\beta/2})\,\mathrm{d}u\to 0 \qquad \text{uniformly in } t \text{ as } \varepsilon\to 0. \tag{13}$$

Let $m\leqslant s(u)\leqslant M$, it is easy to see that for $u\in[\varepsilon,1-\varepsilon]$

$$\left||\sqrt{\varepsilon}\dot{\sigma}_{\varepsilon}^b(u)|^{\beta}-(s(u))^{\beta/2}\right|\leqslant\text{Const}\left|\varepsilon\,[\dot{\sigma}_{\varepsilon}^b(u)]^2-s(u)\right|. \tag{14}$$

Indeed for this last inequality it is enough to prove that $\sqrt{\varepsilon}\,\dot{\sigma}_{\varepsilon}^b(u)$ and $\sqrt{s(u)}$ are bounded above and below on $[\varepsilon,1-\varepsilon]$. It is easy to see that they are bounded above. $\sqrt{s(u)}$ is bounded below by $\sqrt{m}$ and $|\sqrt{\varepsilon}\,\dot{\sigma}_{\varepsilon}^b(u)|>\sqrt{m^2-\varepsilon m^2}>$ Const for $u\in[\varepsilon,1-\varepsilon]$ and (14) holds. Since

$$\begin{aligned}|\varepsilon\,[\dot{\sigma}_{\varepsilon}^b(u)]^2-s(u)|&=\left|\left[\frac{1}{\varepsilon}\int_u^{u+\varepsilon}s(x)\,\mathrm{d}x,-s(u)\right]\left[\int_0^u s(y)\,\mathrm{d}y+\int_{u+\varepsilon}^1 s(y)\,\mathrm{d}y\right]\right.\\&\qquad\left.-s(u)\int_u^{u+\varepsilon}s(y)\,\mathrm{d}y\right|\\&\leqslant M\left|\frac{1}{\varepsilon}\int_u^{u+\varepsilon}s(x)\,\mathrm{d}x-s(u)\right|+M^2\varepsilon\leqslant\text{Const}\,\varepsilon.\end{aligned}$$

This shows (13) and since g verifies the hypothesis (i) and (ii) the Corollary follows. □

Corollary 4. *Under* (H2) *and if* $s\in C^2[0,1]$,

$$\eta_{\varepsilon}(t,x)=\frac{1}{\sqrt{\varepsilon}}\left(\lambda\left\{u\leqslant t,\frac{b^F(u+\varepsilon)-b^F(u)}{\sqrt{\varepsilon}}\leqslant x\right\}-\int_0^t P(\sqrt{s(u)}N\leqslant x)\,\mathrm{d}u\right)$$

converges weakly in $C[0,1]$, *as* $\varepsilon\to 0$ *to* $\tilde{W}(\hat{\sigma}^2(t,x))-K(t,x)$ *where*

$$\begin{aligned}K(t,x)&=b^F(t)\phi\left(\frac{x}{\sqrt{s(t)}}\right)\frac{1}{\sqrt{s(t)}}\\&\qquad-\int_0^t b^F(u)\frac{\dot{s}(u)}{2[s(u)]^{3/2}}\phi\left(\frac{x}{\sqrt{s(u)}}\right)\left[\frac{x^2}{s(u)}-1\right]\mathrm{d}u\end{aligned}$$

and

$$\hat{\sigma}^2(t,x)=2\sum_{n=2}^{\infty}\left[\frac{1}{(n+1)!}\int_0^t H_{n-1}^2\left(\frac{x}{\sqrt{s(u)}}\right)\phi^2\left(\frac{x}{\sqrt{s(u)}}\right)\mathrm{d}u\right].$$

Proof. Let

$$g(u,y)=1_{(-\infty,x]}(uy)-P\{Nu\leqslant x\}+y\phi\left(\frac{x}{u}\right)$$

which satisfies the hypothesis for Theorem 3. Using Theorem 3 we have that

$$\frac{1}{\sqrt{\varepsilon}}\int_0^t g_\varepsilon(u,\xi_\varepsilon^b(u))\,\mathrm{d}u \to \tilde{W}(\hat{\sigma}^2(t,x)) \quad \text{weakly as } \varepsilon\to 0. \tag{15}$$

To finish let us calculate $\sigma^2(t)$ in this case.

$$\begin{aligned} h(u) &= \sum_{k=2}^{+\infty} c_k^2(\sqrt{s(u)})k!\left[\int_{-2}^{+2}\theta^k(w)\,\mathrm{d}w\right] \\ &= \sum_{k=2}^{+\infty}\frac{2}{(k+1)!}H_{k-1}^2\left(\frac{x}{\sqrt{s(u)}}\right)\phi^2\left(\frac{x}{\sqrt{s(u)}}\right), \end{aligned}$$

by (H2) this is uniformly bounded for all $u\in[0,1]$ and so it is continuous in u and it is strictly positive in $[0,1]$. Thus we have shown (15). Now

$$\begin{aligned} \eta_\varepsilon(t,x) &= \frac{1}{\sqrt{\varepsilon}}\int_0^t g_\varepsilon(u,\xi_\varepsilon^b(u))\,\mathrm{d}u - \frac{1}{\sqrt{\varepsilon}}\int_0^t \xi_\varepsilon^b(u)\,\phi\left(\frac{x}{\dot{\sigma}_\varepsilon^b(u)\sqrt{\varepsilon}}\right)\mathrm{d}u \\ &\quad -\frac{1}{\sqrt{\varepsilon}}\int_0^t [P(\sqrt{s(u)}N\leqslant x)-P(N\dot{\sigma}_\varepsilon^b(u)\sqrt{\varepsilon}\leqslant x)]\,\mathrm{d}u = T_1-T_2-T_3. \end{aligned}$$

Let us see first that the last term tends to zero uniformly in t as $\varepsilon\to 0$,

$$\begin{aligned} |T_3| &\leqslant \frac{1}{\sqrt{\varepsilon}}\int_\varepsilon^{1-\varepsilon}[|P(\sqrt{s(u)}N\leqslant x)-P(N\dot{\sigma}_\varepsilon^b(u)\sqrt{\varepsilon}\leqslant x)|]\,\mathrm{d}u \\ &\quad + \frac{1}{\sqrt{\varepsilon}}\left[\int_0^\varepsilon + \int_{1-\varepsilon}^1 [P(\sqrt{s(u)}N\leqslant x)+P(N\dot{\sigma}_\varepsilon^b(u)\sqrt{\varepsilon}\leqslant x)]\,\mathrm{d}u\right]. \end{aligned} \tag{16}$$

Since $\varepsilon[\dot{\sigma}_\varepsilon^b(u)]^2$ and $s(u)$ are bounded below and $|\varepsilon[\dot{\sigma}_\varepsilon^b(u)]^2-s(u)|\leqslant \text{Const}\,\varepsilon$ for $u\in[\varepsilon,1-\varepsilon]$, the term $\left|P(\sqrt{s(u)}N\leqslant x)-P(N\dot{\sigma}_\varepsilon^b(u)\sqrt{\varepsilon}\leqslant x)\right|$ is bounded above by Const ε. Thus we have that (16) $\leqslant \text{Const}\sqrt{\varepsilon}$.

We observe that T_2 is tight, since the function

$$f_\varepsilon(u,y)=y\phi\left(\frac{x}{\sqrt{\varepsilon}\dot{\sigma}_\varepsilon^b(u)}\right)$$

satisfies all the hypothesis except for $E(Nf_\varepsilon(u,N))=0$, which is not necessary to prove tightness. Integrating by parts

$$\begin{aligned} T_2 &= \int_0^t \dot{b}_\varepsilon^F(u)\,\frac{1}{\dot{\sigma}_\varepsilon^b(u)\sqrt{\varepsilon}}\,\phi\left(\frac{x}{\dot{\sigma}_\varepsilon^b(u)\sqrt{\varepsilon}}\right)\mathrm{d}u = b_\varepsilon^F(t)\,\frac{1}{\dot{\sigma}_\varepsilon^b(t)\sqrt{\varepsilon}}\,\phi\left(\frac{x}{\dot{\sigma}_\varepsilon^b(t)\sqrt{\varepsilon}}\right) \\ &\quad - b_\varepsilon^F(0)\frac{1}{\dot{\sigma}_\varepsilon^b(0)\sqrt{\varepsilon}}\phi\left(\frac{x}{\dot{\sigma}_\varepsilon^b(0)\sqrt{\varepsilon}}\right) \\ &\quad - \int_0^t b_\varepsilon^F(u)\frac{[\sqrt{\varepsilon}\dot{\sigma}_\varepsilon^b(u)]'}{\varepsilon[\dot{\sigma}_\varepsilon^b(u)]^2}\phi\left(\frac{x}{\dot{\sigma}_\varepsilon^b(u)\sqrt{\varepsilon}}\right)\left[\frac{x^2}{\varepsilon[\dot{\sigma}_\varepsilon^b(u)]^2}-1\right]\mathrm{d}u \end{aligned}$$

(here $'$ denotes the derivative) with

$$[\sqrt{\varepsilon}\,\dot{\sigma}^b_\varepsilon(u)]' = \frac{\frac{1}{\varepsilon}[s(u+\varepsilon)-s(u)][F(u)+1-F(u+\varepsilon)]+\frac{1}{\varepsilon}[F(u+\varepsilon)-F(u)][s(u)-s(u+\varepsilon)]}{2\sqrt{\varepsilon}\dot{\sigma}^b_\varepsilon(u)}$$

for $u \in [0, 1-\varepsilon]$, hence

$$\lim_{\varepsilon\to 0}[\sqrt{\varepsilon}\,\dot{\sigma}^b_\varepsilon(u)]' = \frac{\dot{s}(u)}{2\sqrt{s(u)}} \quad \text{for } u \in [0,1).$$

To finish the proof of the corollary it remains to show that $T_1 - T_2$ converges weakly in finite-dimensional distributions, as $\varepsilon \to 0$, to

$$\tilde{W}(\hat{\sigma}^2(t,x)) - b^F(t)\phi\left(\frac{x}{\sqrt{s(t)}}\right)\frac{1}{\sqrt{s(t)}} + \int_0^t b^F(u)\frac{\dot{s}(u)}{2[s(u)]^{3/2}}\phi\left(\frac{x}{\sqrt{s(u)}}\right)\left[\frac{x^2}{s(u)}-1\right]\mathrm{d}u.$$

This can be done using a discretization procedure and the convergence in law of $(S^b_\varepsilon(t), b^F_\varepsilon(t'))$ towards $(\tilde{W}(\hat{\sigma}^2(t,x)), b^F(t'))$. $\square$

4.2. Estimation of a diffusion's variance

In this section we shall consider the problem of estimating the variance of a non-homogeneous diffusion, which has been considered before by Genon-Catalot et al. (1992) and Soulier (1998). They considered a discretization of the diffusion instead of regularization by convolution, as we do.

By Girsanov's theorem the solution to the stochastic differential equation

$$\mathrm{d}X(t) = \vartheta(t)\,\mathrm{d}W(t) + b(X(t))\,\mathrm{d}t \quad \vartheta(t) > 0$$

induces a measure which is absolutely continuous with respect to the measure associated to the solution of $\mathrm{d}X(t) = \vartheta(t)\mathrm{d}W(t)$, which is a BM with a time change. Note that $s(t) = \vartheta^2(t)$ but now the integral of ϑ^2 is not necessarily equal to one. Hence we shall use the notation $X(t) := W^\vartheta(t)$. We assume that ϑ is continuous and that we observe a regularization of the diffusion ($W^\vartheta_\varepsilon(t)$ defined below) on a compact interval which does not contain 0, hence $\vartheta(t)$ is bounded uniformly away from 0. This is to guarantee the uniform convergence of

$$\frac{\sqrt{\varepsilon}}{\|\varphi\|_2}\dot{\sigma}_\varepsilon(t-hv) \text{ to } \vartheta(t)$$

(for $t = 0$ this expression converges to $\vartheta(t)/2$). By previous considerations about stable convergence and without loss of generality, we can work with this diffusion. Let, then,

$$W^\vartheta_\varepsilon(t) = \frac{1}{\varepsilon}\int_{-\infty}^{\infty}\varphi\left(\frac{t-s}{\varepsilon}\right)W^\vartheta(s)\,\mathrm{d}s,$$

where φ is even, let K be a centred probability density with support in $[-1,1]$ and N a standard Gaussian r.v. We define an estimator of $\alpha(t)=EG(\vartheta(t)N)$, with $h(\varepsilon)\to 0$ with $\varepsilon\to 0$, and $G\in L^2(\phi(x)\,\mathrm{d}x)$ an even continuous function by

$$\hat{\alpha}_\varepsilon(t)=\frac{1}{h}\int_{t-h}^{t+h}K\left(\frac{t-u}{h}\right)G\left(\frac{\sqrt{\varepsilon}}{\|\varphi\|_2}\dot{W}_\varepsilon^{\vartheta}(u)\right)\mathrm{d}u.$$

Interesting particular cases of G are:

(1) $G(x)=x^2$, $\alpha(t)=\vartheta^2(t)$,

(2) $G(x)=\sqrt{(\frac{\pi}{2})}|x|$, $\alpha(t)=\vartheta(t)$,

(3) $G(x)=\log(|x|)-2\gamma$, with $\gamma=\int_0^\infty\log(x)\phi(x)\,\mathrm{d}x$, then $\alpha(t)=\log(\vartheta(t))$.

These estimators are L^2 consistent. To see this let

$$E\{\hat{\alpha}_\varepsilon(t)\}=\frac{1}{h}\int_{t-h}^{t+h}K\left(\frac{t-u}{h}\right)E\left\{G\left(\frac{\sqrt{\varepsilon}}{\|\varphi\|_2}\dot{\sigma}_\varepsilon(u)N\right)\right\}\mathrm{d}u$$

$$=\int_{-\infty}^{\infty}K(v)E\left\{G\left(\frac{\sqrt{\varepsilon}}{\|\varphi\|_2}\dot{\sigma}_\varepsilon(t-hv)N\right)\right\}\mathrm{d}v.$$

Remember that $\dot{\sigma}_\varepsilon(u)$ is given by (1). Since the sequence $(\sqrt{\varepsilon}/\|\varphi\|_2)\dot{\sigma}_\varepsilon(t-hv)$ converges uniformly to $\vartheta(t)$, we have

$$E\{\hat{\alpha}_\varepsilon(t)\}\to\alpha(t).$$

Let now $g(x,y)=G(xy)-E[G(xN)]$, and $g_\varepsilon(u,y)=g((\sqrt{\varepsilon}/\|\varphi\|_2)\dot{\sigma}_\varepsilon(u),y)$; since G is even we have that $E[g(x,N)]=0$ and $E[g(x,N)N]=0$. On the other hand it is easy to see that $g_\varepsilon(u,y)\to g(\vartheta(u),y)$. Let us look now at

$$E(\hat{\alpha}_\varepsilon(t)-\alpha(t))^2=E\left(\frac{1}{h}\int_{-\infty}^{\infty}K\left(\frac{t-u}{h}\right)g_\varepsilon(u,\zeta_\varepsilon^W(u))\,\mathrm{d}u\right)^2+(E(\hat{\alpha}_\varepsilon(t))-\alpha(t))^2$$

Consider the first term.

$$\frac{h}{\varepsilon}E\left(\frac{1}{h}\int_{-\infty}^{\infty}K\left(\frac{t-u}{h}\right)g_\varepsilon(u,\zeta_\varepsilon^W(u))\,\mathrm{d}u\right)^2$$

$$=\frac{1}{\varepsilon h}\int_{R^2}K\left(\frac{t-u}{h}\right)K\left(\frac{t-u'}{h}\right)E[g_\varepsilon(u,\zeta_\varepsilon^W(u))g_\varepsilon(u',\zeta_\varepsilon^W(u'))]\,\mathrm{d}u\,\mathrm{d}u']=(I).$$

Now the Hermite expansion of $g_\varepsilon(u,y)$ is

$$g_\varepsilon(u,y)=\sum_{k=1}^{\infty}c_{2k,\varepsilon}(u)H_{2k}(y)$$

with

$$c_{2k,\varepsilon}(u)\to c_{2k}(u)=\frac{1}{(2k)!}\int_{-\infty}^{\infty}g(\vartheta(u),x)H_{2k}(x)\phi(x)\,\mathrm{d}x.$$

By Mehler's formula

$$(I)=\frac{1}{\varepsilon h}\int_{R^2}K\left(\frac{t-u}{h}\right)K\left(\frac{t-u'}{h}\right)\sum_{k=1}^{\infty}c_{2k,\varepsilon}(u)c_{2k,\varepsilon}(u')(2k)!$$

$$\times\left[\frac{1}{\varepsilon\dot{\sigma}_\varepsilon(u)\dot{\sigma}_\varepsilon(u')}\int_{-\infty}^{\infty}\varphi(w)\varphi\left(\frac{u-u'}{\varepsilon}+w\right)\vartheta^2(u-\varepsilon w)\,\mathrm{d}w\right]^{2k}\mathrm{d}u\,\mathrm{d}u'.$$

Making the change of variables $u' = u + \varepsilon z$ and $t - u = hv$ we get

$$(I) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} K(v)K\left(v - \frac{\varepsilon}{h}z\right)\sum_{k=1}^{\infty} c_{2k,\varepsilon}(t-hv)c_{2k,\varepsilon}(t-hv+\varepsilon z)(2k)!$$

$$\times\left[\frac{1}{\varepsilon\dot{\sigma}_\varepsilon(t-hv)\dot{\sigma}_\varepsilon(t-hv+\varepsilon z)}\int_{-\infty}^{\infty}\varphi(w)\varphi(w-z)\vartheta^2(t-hv-\varepsilon w)\,\mathrm{d}w\right]^{2k}\mathrm{d}v\,\mathrm{d}z.$$

Observing that φ and K have support in $[-1,1]$, if $\varepsilon = \mathrm{o}(h(\varepsilon))$ this expression converges, as ε goes to zero, to

$$\left[\int_{-1}^{1} K^2(v)\,\mathrm{d}v\right]\sum_{k=1}^{\infty} c_{2k}^2(t)(2k)!\int_{-2}^{+2}\theta^{2k}(w)\,\mathrm{d}w = (II).$$

We have thus shown that

$$E\left(\frac{1}{h}\int_{-\infty}^{\infty} K\left(\frac{t-u}{h}\right)g_\varepsilon(u,\zeta_\varepsilon^W(u))\,\mathrm{d}u\right)^2 = \mathrm{O}\left(\frac{\varepsilon}{h}\right).$$

Let us calculate the value of (II) for the three examples of G given above.

(1) $G(x) = x^2$

$$(II) = \int_{-1}^{1} K^2(v)\,\mathrm{d}v 2\vartheta^4(t)\int_{-2}^{+2}\theta^2(w)\,\mathrm{d}w.$$

(2) $G(x) = \sqrt{\frac{\pi}{2}}|x|$

$$(II) = \int_{-1}^{1} K^2(v)\,\mathrm{d}v\,\vartheta^2(t)\int_{-2}^{+2}(\sqrt{1-\theta^2(w)} + \theta(w)\arcsin(\theta(w)) - 1)\,\mathrm{d}w.$$

(3) $G(x) = \log(|x|) - 2\gamma$

$$(II) = \int_{-1}^{1} K^2(v)\,\mathrm{d}v\int_{-2}^{+2} E((\log(|X|)\log(|\theta(w)X + \sqrt{1-\theta^2(w)}Y|)) - 4\gamma^2)\,\mathrm{d}w.$$

where X and Y are independent standard Gaussian variables.

It is important to observe that this last variance is independent of ϑ. By the previous calculation for the variance we see that the estimator is L^2 consistent. To establish the balance between variance and bias we have to obtain the speed with which the bias goes to zero, and for this we have to impose a regularity condition on $\vartheta^2(t)$. We shall assume that it has two continuous derivatives. We have then

$$E[\hat{\alpha}_\varepsilon(t)] - \alpha(t) = \int_{-\infty}^{\infty} E\left[G\left(\frac{\sqrt{\varepsilon}}{\|\varphi\|_2}\dot{\sigma}_\varepsilon(t-hv)N\right)\right] - E[G(\vartheta(t)N)]K(v)\,\mathrm{d}v.$$

Using the Mean Value Theorem for G in the first case and the Lipschitz property in the other two, the fact that $\vartheta(t)$ is bounded below (since it is continuous, strictly positive and is observed in a compact set) and Taylor's Theorem for $(\varepsilon/\|\varphi\|_2^2)\dot{\sigma}_\varepsilon^2(t-hv) - \vartheta^2(t)$ we get that

$$\frac{1}{h^2}E[\hat{\alpha}_\varepsilon(t) - \alpha(t)] \to a_i(t) i = 1,2,3,$$

$$a_1(t) = \frac{[\vartheta^2]''(t)}{2}\int_{-\infty}^{\infty} K(v)v^2\,\mathrm{d}v, \qquad a_2(t) = \frac{[\vartheta^2]''(t)}{4\vartheta(t)}\int_{-\infty}^{\infty} K(v)v^2\,\mathrm{d}v,$$

$$a_3(t) = \frac{[\vartheta^2]''(t)}{4\vartheta^2(t)}\int_{-\infty}^{\infty} K(v)v^2\,\mathrm{d}v.$$

To balance out the bias and variance terms we have to choose $h=\varepsilon^{1/5}$, which we shall call the optimal choice. With this we define

$$\beta_\varepsilon(t)=\frac{1}{\varepsilon^{2/5}}\int_{-1}^{1}K(v)g_\varepsilon(t-\varepsilon^{1/5}v,\zeta_\varepsilon^W(t-\varepsilon^{1/5}v))\,\mathrm{d}v.$$

Then $E[\beta_\varepsilon(t)]=0$ and also $E[\beta_\varepsilon^2(t)]\to\int_{-1}^{1}K^2(v)\,\mathrm{d}v\sum_{k=1}^{\infty}c_{2k}^2(t)(2k)!\int_{-2}^{+2}\theta^{2k}(w)\,\mathrm{d}w$. Letting $N=[1/\varepsilon^{4/5}]$ then as before

$$\beta_\varepsilon(t)\approx\frac{1}{\varepsilon^{2/5}}\sum_{j=-N}^{N-1}\int_{j\varepsilon^{4/5}}^{(j+1)\varepsilon^{4/5}}g_\varepsilon(t-\varepsilon^{1/5}v,\zeta_\varepsilon^W(t-\varepsilon^{1/5}v))K(v)\,\mathrm{d}v.$$

This expression is the sum of 1-dependent r.v.'s so the Central Limit Theorem holds. As before the limit Gaussian r.v. is independent of $\mathscr{F}$ and the convergence is stable. Thus $h=\varepsilon^{1/5}$

$$\varepsilon^{-2/5}[\hat{\alpha}_\varepsilon(t)-\alpha(t)]$$
$$\to N\left(a_i(t),\int_{-1}^{1}K^2(v)\,\mathrm{d}v\sum_{k=1}^{\infty}c_{2k}^2(t)(2k)!\int_{-2}^{+2}\theta^{2k}(w)\,\mathrm{d}w+a_i^2(t)\right),$$

where the $a_i(t)$ depend on the function G chosen. □

This result can be applied to more general diffusions processes with $b(t)\neq 0$, using the fact that stable convergence is invariant under absolutely continuous changes of measure. Also it is possible to prove a theorem for G satisfying appropriate regularity conditions. Observe that these results can be readily extended to the solution of stochastic differential equations driven by the Brownian bridge, by using the properties of stable convergence.

4.3. Crossings and local time

Corollary 5. *Under* (H1), (H2), *and* $f\in C^2$ *with* $\ddot{f}$ *bounded*, $Z_\varepsilon^b(f)$ *converges stably as* $\varepsilon\to 0$ *towards a r.v.* $V=\sigma_1\int_0^1 f(b^F(u))\sqrt{s(u)}\,\mathrm{d}\tilde{W}(u)$.

Proof. Recall that

$$Z_\varepsilon^{b^F}(f)=\varepsilon^{-1/2}\int_{-\infty}^{\infty}f(x)[\Lambda_\varepsilon N_\varepsilon^{b^F}(x)-\tilde{\ell}^{b^F}(x)]\,\mathrm{d}x\quad\text{with } \Lambda_\varepsilon^{-1}=\sqrt{\frac{2}{\pi\varepsilon}}\|\varphi\|_2.$$

To simplify the notation we shall call $k_\varepsilon(u)=\sqrt{\varepsilon}\dot{\sigma}_\varepsilon^b(u)\|\varphi\|_2^{-1}$. Using Banach's formula (Banach, 1925)

$$\int_{-\infty}^{+\infty}f(x)N_\varepsilon^b(x)\,\mathrm{d}x=\int_0^1 f(b_\varepsilon^F(u))|\dot{b}_\varepsilon^F(u)|\,\mathrm{d}u$$

we have the decomposition of $Z_\varepsilon^b(f)$:

$$Z_\varepsilon^b(f)=\frac{1}{\sqrt{\varepsilon}}\int_0^1 f(b_\varepsilon^F(u))g_\varepsilon(u,\xi_\varepsilon^b(u))\,\mathrm{d}u+\frac{1}{\sqrt{\varepsilon}}\int_0^1[f(b_\varepsilon^F(u))-f(b^F(u))]k_\varepsilon(u)\,\mathrm{d}u$$
$$+\frac{1}{\sqrt{\varepsilon}}\int_0^1 f(b^F(u))[k_\varepsilon(u)-\sqrt{s(u)}]\,\mathrm{d}u=T_1+T_2+T_3$$

where $g_\varepsilon(u,y)=l(y)k_\varepsilon(u)$ and $l(y)=\sqrt{\frac{\pi}{2}}|y|-1$.

A proof along the lines of Berzin-Joseph and León (1997) shows that T_2 and T_3 converge to zero in probability and that T_1 converges to V. Note that stable convergence holds under absolutely continuous changes of the measure.

Acknowledgements

The authors would like to thank the referee for many suggestions that led to substantial improvements in this work.

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